

Thermal Fluctuations and Bouncing Cosmologies

Yi-Fu Cai^{1*}, Wei Xue^{2†}, Robert Brandenberger^{2,1,3,4‡}, Xinmin Zhang^{1,3§}

1) *Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, P. R. China*

2) *Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada*

3) *Theoretical Physics Center for Science Facilities (TPCSF), CAS, P. R. China* and

4) *Kavli Institute for Theoretical Physics, CAS, Beijing 100190, P. R. China*

We study the conditions under which thermal fluctuations generated in the contracting phase of a non-singular bouncing cosmology can lead to a scale-invariant spectrum of cosmological fluctuations at late times in the expanding phase. We consider point particle gases, holographic gases and string gases. In the models thus identified, we also study the thermal non-Gaussianities of the resulting distribution of inhomogeneities. For regular point particle radiation, we find that the background must have an equation of state $w = 7/3$ in order to obtain a scale-invariant spectrum, and that the non-Gaussianities are suppressed on scales larger than the thermal wavelength. For Gibbons-Hawking radiation, we find that a matter-dominated background yields scale-invariance, and that the non-Gaussianities are large. String gases are also briefly considered.

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I. INTRODUCTION

In recent years, there has been a lot of interest in non-singular bouncing cosmologies (see e.g [1] for a recent review with an extensive list of references). Such cosmologies may be desirable since they resolve the singularity problem of the inflationary scenario, the current paradigm of early universe cosmology. If inflation is realized by making use of the potential energy of a scalar matter field while treating space-time dynamics using the Einstein action, then an initial cosmological singularity is unavoidable [2].

While it is possible that that in the context of a non-singular bouncing cosmology a period of inflationary expansion [3] is realized after the bounce (see e.g. [4] for an explicit model) and the cosmological perturbations observed today are generated as quantum vacuum fluctuations in the inflationary phase [5], it is also possible to obtain fluctuations without requiring an inflationary phase after the bounce. The reason is that all scales observed today were at one point inside the Hubble radius at sufficiently early times during the contracting phase. This is illustrated in Figure 1, which shows the space-time plot in a non-singular bouncing cosmology [43].

In the context of studies of bouncing cosmologies it has been realized that fluctuations which are generated as quantum vacuum perturbations and exit the Hubble radius during a matter-dominated contracting phase lead to a scale-invariant spectrum of cosmological fluctuations today [11, 12, 13] (see also [14]), and thus yield an alternative to cosmological inflation for explaining the current observational data.

However, in a bouncing cosmology it is not manifest that perturbations arise as quantum vacuum fluctuations. In inflationary cosmology it can be argued that the exponential expansion of space during the inflationary phase red-shifts all classical matter initially present and leaves behind a matter vacuum. However, if the universe starts out large and cold in a contracting phase, there does not seem to be a reason to single out vacuum over thermal initial conditions for the fluctuations.

Thermal fluctuations as the origin of structure in the universe were considered in the context of a forever expanding cosmology, but it was concluded that it was not possible to obtain a scale-invariant spectrum of cosmological perturbations [15]. However, as realized in [7], thermal fluctuations of strings can give rise to a scale-invariant spectrum if we abandon the assumption that the universe is forever expanding. Specifically, in [7] it was shown that thermal string fluctuations in a quasi-static early string phase yield a scale-invariant spectrum. Hence, there are good reasons to expect that in a bouncing cosmology it might be possible for thermal fluctuations to generate a spectrum whose shape is in good agreement with observations.

In this paper we will study the conditions under which thermal initial conditions for perturbations can lead to a scale-invariant spectrum of cosmological fluctuations after the bounce [44]. We find various possibilities, depending on what kind of thermal fluctuations we consider

If the fluctuations are generated by normal particle radiation (with the usual equation of state $w_r = 1/3$, where w is the ratio $w = p/\rho$ of pressure p to energy density ρ), then the required equation of state of the background is $w = 7/3$. For radiation in holographic cosmology [18], we find that a background equation of state of $w = 0$ will lead to a scale-invariant spectrum. String gases require a quasi-static early phase.

As realized [19] in the study of perturbations in the “matter bounce”, in which vacuum fluctuations exit the Hubble radius during a matter-dominated phase of con-

*cai@ihep.ac.cn

†xuewei@physics.mcgill.ca

‡rhh@physics.mcgill.ca

§xmzhang@ihep.ac.cn

traction, inhomogeneities generated during a contracting phase leave a distinctive imprint on the magnitude and shape of the non-Gaussianities manifest in the three point function. We will study these non-Gaussianities for thermal initial conditions.

The outline of this paper is as follows: In the following section we study the evolution of fluctuations in a bouncing universe. In Section 3, we consider thermal initial conditions for the perturbations and ask under which conditions a scale-invariant spectrum after the bounce results. In Section 4 we then estimate the non-Gaussianities in the resulting models.

II. FLUCTUATIONS IN A BOUNCING UNIVERSE

It is useful to take a first glance at the route of cosmological perturbations in a bouncing cosmology. As is depicted in Figure 1, the wavelength of fluctuations becomes larger than the Hubble radius in the contracting phase, and reenters in the expanding phase. All that is required for this space-time sketch to apply is that the equation of state parameter w of the background universe be larger than $-1/3$ at all times except possibly around the bounce point. This requirement is automatically satisfied for most forms of matter (the exception being scalar field models which lead to inflation). In Figure 1 we plot the evolution of the physical length corresponding to a fixed comoving scale. This scale is the wavelength of the fluctuation mode k which we will follow later.

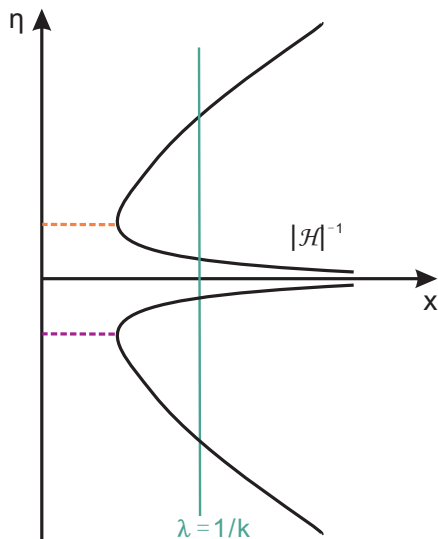


FIG. 1: A sketch of the evolution of scales in a bouncing universe. The horizontal axis is the physical spatial coordinate, the vertical axis is time. Plotted are the comoving Hubble radius $|\mathcal{H}|^{-1}$ and the wavelength λ of a fluctuation mode with fixed comoving wavenumber k .

In this paper we focus on adiabatic fluctuations and consider matter without anisotropic stress. In this

case, the linearized fluctuations about a Friedmann-Robertson-Walker background metric in longitudinal gauge can be expressed as (see e.g. [20] for a review of the theory of cosmological perturbations)

$$ds^2 = a(\eta)^2[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)d\vec{x}^2], \quad (1)$$

where $\Phi(\vec{x}, \eta)$ is the generalized Newtonian gravitational potential which characterizes the metric fluctuations, η is conformal time and $a(\eta)$ is the background scale factor [45].

In Fourier space, the fluctuation variable Φ satisfies the following second order differential equation

$$\Phi_k'' + 2\sigma\mathcal{H}\Phi_k' + (c_s^2k^2 - 2\epsilon\mathcal{H}^2 + 2\sigma\mathcal{H}^2)\Phi_k = 0, \quad (2)$$

where $\mathcal{H} \equiv a'/a$ is the comoving Hubble parameter, and the prime denotes the derivative with respect to η . The parameter c_s is the sound speed, which we take to be a free parameter. Moreover, we have defined two useful parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \text{ and} \quad (3)$$

$$\sigma \equiv -\frac{\ddot{H}}{2H\dot{H}} \quad (4)$$

which characterize the background evolution. For a constant equation of state w , we have

$$\epsilon = \sigma = \frac{3}{2}(1 + w) \quad (5)$$

and thus the perturbation equation can be greatly simplified.

We will consider an equation of state $w \neq -1$ for which the background scale factor evolves as a power of time. If $w > -1/3$, an equation of state which does not give accelerated expansion, we can choose our origin of the time axis such that $t = 0$ corresponds to the bounce point. More specifically, we shall take $t = 0$ to be the value of t for which the Big Crunch singularity would occur in the absence of the terms in the action which lead to a non-singular bounce. In this case, we can also choose the origin of the conformal time coordinate such that $t = 0$ corresponds to $\eta = 0$. With this choice of coordinates, then in the contracting phase, the perturbation equation (2) takes the form

$$\Phi_k'' + \frac{1 + 2\nu}{\eta}\Phi_k' + c_s^2k^2\Phi_k = 0, \quad (6)$$

where

$$\nu \equiv \frac{5 + 3w}{2(1 + 3w)}. \quad (7)$$

Since (6) is a second order differential equation, there are two linearly independent solutions. On super-Hubble scales, one mode is constant (the ‘‘D-mode’’), the other is

growing in a contracting phase (and decreasing in an expanding phase). We call the second mode the “S-mode”. The general solution on super-Hubble scales is a linear combination of the two modes and is hence given by

$$\Phi_k^- = D_- + \frac{S_-}{(\eta)^{2\nu}}, \quad (8)$$

where D_- and S_- are the mode coefficients. If the equation of state is the same in the expanding phase after the non-singular bounce, then the solution in the expanding period can be written as

$$\Phi_k^+ = D_+ + \frac{S_+}{(\eta)^{2\nu}} \quad (9)$$

with new mode coefficients D_+ and S_+ . In a non-singular bounce, the fluctuations can be smoothly evolved through the bounce. Thus, the mixing matrix which relates the mode coefficients in the expanding phase with those in the contracting phase can be calculated [46]

The key question which arises when studying the transfer of fluctuations through a non-singular bounce is whether the spectrum of the dominant mode in the contracting phase, the S_- mode, couples to the D_+ mode, the dominant mode in the expanding phase. This issue was initially studied by replacing the bounce region with a matching surface across which the perturbations are connected making use of the Hwang-Vishniac [23] (Deruelle-Mukhanov [24]) matching conditions. It was found [25, 26, 27] that the coupling is suppressed by a factor of k^2 on large wavelengths, i.e.

$$D_+ = \mathcal{O}(1)D_- + \mathcal{O}(1)\left(\frac{k}{k^*}\right)^2 S_- \quad (10)$$

where k^* is a normalization scale which is set by the microphysics, i.e. is in the ultraviolet range. However, as pointed out in [28], it is not valid to apply the matching conditions to fluctuations at an interface between a contracting and an expanding phase because the background does not satisfy the matching conditions. If the bounce is non-singular, however, one does not need to make use of matching conditions: the perturbations can be evolved continuously from the contracting to the expanding phase. This was done in the context of a non-singular regularization of Pre-Big-Bang cosmology in [29], and in the context of a non-singular Ekpyrotic model in [30]. More recently, such calculations were carried out for a non-singular mirage cosmology [31], in the non-singular higher derivative bounce of [16] in [32], for a quantum bounce model in [33], and more specifically in the Lee-Wick bounce in [22].

For a non-singular bounce, one can follow the fluctuations both numerically and analytically. To obtain a good analytical approximation, one divides the background time into three intervals - the contracting phase, the bounce phase where the Hubble expansion rate can be modelled as $H(t) = \alpha t$ (where α is a constant whose value

is set by the new physics which determines the bounce), and the post-bounce expanding phase. The duration of the bounce phase is set by the scale determining the new physics which regulates the bounce. The result of the works quoted above is that on length scales larger than the duration of the bounce, the mode mixing occurs as given by (10). Thus, the contribution of the dominant mode in the contracting phase is suppressed in the dominant expanding phase mode function by a factor of k^2 . On the other hand, on scales short compared the bounce time, there is no suppression of the coupling.

In the following, we will, without much loss of generality, assume that the bounce is short from the point of view of cosmological scales of interest. Thus, we will use the coupling given by (10). In [22], we studied the evolution of perturbations which start out as quantum vacuum fluctuations and discovered that it is precisely quantum vacuum fluctuations which exit the Hubble radius is a matter-dominated contracting phase which are scale-invariant after the bounce. However, in the framework of a cosmological model which starts out large and cold, there is no particular reason to focus on quantum vacuum initial fluctuations. It is rather reasonable to consider initial thermal perturbations. In this paper we will study under which conditions on the background cosmology one obtains a scale-invariant spectrum at later times starting from thermal initial conditions.

Before starting the analysis, we remind the reader of a useful formula for the time of Hubble radius crossing. Given an equation of state parameter w , the scale factor evolves as

$$a(t) \sim t^p, \quad \text{with } p = \frac{2}{3(1+w)}, \quad (11)$$

and yields the following relation for the conformal time η

$$\eta \sim t^{1-p}. \quad (12)$$

The condition for Hubble radius crossing is

$$k \sim aH \quad (13)$$

for a perturbation mode with fixed comoving wavenumber k . This yields

$$\eta_H(k) \sim k^{-1}, \quad (14)$$

and thus

$$t_H(k) \sim k^{-\frac{1}{1-p}}, \quad (15)$$

where the subscript “ H ” denotes the moment of Hubble radius crossing.

III. THERMAL FLUCTUATIONS

The method of calculating the spectrum of cosmological perturbations at late times is the following. First, we

compute the matter fluctuations on sub-Hubble scales in the contracting phase. Next, for any scale k , we compute the induced metric fluctuations at the time $t_H(k)$ when the scale exits the Hubble radius during the contracting phase. In the third step, the metric fluctuations are evolved on super-Hubble scales making use of the evolution equations for perturbations discussed in the previous section. This is the standard way of following the generation and evolution of cosmological perturbations, as applied to inflationary cosmology in early works (see e.g. [34] and [35]) and to string gas cosmology [7, 9, 10]. The method of calculation reflects the fact that metric fluctuations are sub-dominant on sub-Hubble scales, but that on super-Hubble scales the matter fluctuations freeze out and the evolution of the perturbations is driven by the metric.

The key constraint equation which relates matter and metric fluctuations is the time-time component of the perturbed Einstein equation

$$-3\mathcal{H}(\mathcal{H}\Phi + \Phi') + \nabla^2\Phi = 4\pi G a^2 \delta\rho, \quad (16)$$

where $\delta\rho$ is the fluctuation of the energy density. Note that all the three terms on the left-hand-side of the above equation are of the same order of magnitude at the Hubble radius crossing time. Therefore, up to a constant of order $O(1)$, the power spectrum of the metric perturbations is given by,

$$\begin{aligned} P_\Phi(k) &\equiv \frac{1}{12\pi^2} k^3 |\Phi(k)|^2 \\ &= \frac{1}{4M_p^4} k^3 \frac{\langle \delta\rho(k)^2 \rangle}{(H(t_H(k)))^4}, \end{aligned} \quad (17)$$

where (in our case) the pointed brackets denote ensemble averaging in thermal equilibrium. Making use of the Hubble radius crossing condition $H(t_H(k)) = a^{-1}(t_H(k))k$ and replacing the power spectrum of the Fourier space energy density correlation function by the position space correlation function we obtain:

$$P_\Phi(k)(t_H(k)) = \frac{1}{4M_p^4} k^{-4} a^4(t_H(k)) \langle \delta\rho^2 \rangle_{|R(k)}, \quad (18)$$

where $\langle \delta\rho^2 \rangle_{|R(k)}$ (to be evaluated at Hubble radius crossing) is the position space energy density fluctuation correlation function is a sphere of radius $R(k)$, where $R(k)$ is the physical length corresponding to the co-moving momentum scale k .

In a system which is in thermal equilibrium, the correlation function of the energy density is given by

$$\langle \delta\rho^2 \rangle_{|R(k)} \equiv C_V(R) \frac{T^2}{R^6}, \quad (19)$$

where $C_V(R)$ is the heat capacity in a sphere of radius R and is defined in terms of the expectation value of the internal energy as

$$C_V(R) \equiv \frac{\partial}{\partial T} \langle E \rangle. \quad (20)$$

A. Thermal Particle Fluctuations

In this subsection we consider fluctuations in a gas of point particles with an arbitrary equation-of-state w_r . In this case, from the stress-energy conservation equation it follows that the energy density and the temperature change as a function of the scale factor as follows:

$$\rho_r \sim a^{-3(1+3w_r)} \quad (21)$$

$$T \sim a^{-3w_r}, \quad (22)$$

which yields

$$\rho_r \sim T^{1+\frac{1}{w_r}}, \quad (23)$$

and so we get the heat capacity

$$C_V(R) = R^3 \frac{\partial \rho}{\partial T} \sim R^3 T^{\frac{1}{w_r}}. \quad (24)$$

Inserting (19) and (24) into (17) and applying $R \sim 1/H$, we obtain the power spectrum for metric perturbations at the Hubble radius crossing time $t_H(k)$:

$$P_\Phi(k) \sim T_H^{2+\frac{1}{w_r}} H_{t_H(k)}^{-1} \sim k^{\frac{1-3p(1+2w_r)}{p-1}}, \quad (25)$$

where we have used the relations

$$T_H(k) \sim a^{-3w_r} \sim k^{\frac{3w_r p}{1-p}}, \text{ and} \quad (26)$$

$$H_{t_H(k)} \sim k^{\frac{1}{1-p}}. \quad (27)$$

Since the power spectrum of the constant mode $P_D(k)$ is the same as that of Φ at Hubble radius crossing, it scales as

$$P_D(k) \sim k^{n_D} \quad (28)$$

$$n_D = \frac{1-3p(1+2w_r)}{p-1}.$$

The power spectrum of the growing mode $P_S(k)$ is the spectrum of Φ at Hubble radius crossing modulated by the factor $\eta_H(k)^{4\nu}$. This yields

$$P_S(k) \sim P_D(k) k^{-4\nu} \sim k^{n_S} \quad (29)$$

$$n_S = \frac{3-p(1+6w_r)}{p-1}.$$

There are two possibilities to obtain a scale-invariant spectrum after the bounce. The first is if the D-mode in the contracting phase is scale-invariant (recall that the contribution of the contracting phase D-mode to the expanding phase D-mode is not suppressed). The second is that the S-mode in the contracting phase has a spectrum proportional to k^{-4} . Since the contribution of the contracting phase S-mode to the expanding phase D-mode is suppressed by k^4 in the power spectrum (see (10), this then yields scale-invariance of the dominant mode in the

expanding phase. Thus, the two possibilities to obtain a scale-invariant spectrum are

$$n_D = 0, \quad n_S + 4 > 0; \quad (30)$$

$$\text{or } n_D > 0, \quad n_S + 4 = 0. \quad (31)$$

In order for the D-mode to provide a scale-invariant spectrum, we require

$$w_r = \frac{1}{4}(w - 1). \quad (32)$$

This is a solution provided that the S-mode yields a blue spectrum which requires

$$\frac{w + 4w_r - 1}{w + 1/3} \geq 0 \quad (33)$$

which is satisfied if $w > 1$. Note that for such a background, the anisotropic stress grows less fast than the background energy density. Thus, this background is stable towards anisotropic stress fluctuations. Note also that if we were to demand that the equation of state of the background and of the fluctuations is the same, we would obtain $w = -1/3$ which is not a physical solution since it is the borderline solution between an accelerating and a decelerating background, and no scales exit the Hubble radius during the contracting phase.

For the growing mode (S-mode) in the contracting phase to provide a scale-invariant spectrum in the expanding phase, the condition on the two equations of state is

$$w_r = \frac{1}{4}(w - 1), \quad (34)$$

and the dominance of the S-mode requires

$$-\frac{1}{3} < w < 1. \quad (35)$$

In the special case that the radiation is normal radiation with $w_r = \frac{1}{3}$, the only possibility to obtain a scale-invariant spectrum is via the D-mode in the contracting phase. This requires

$$w = \frac{7}{3}. \quad (36)$$

B. Fluctuations in Holographic Cosmology

Gibbons-Hawking radiation [36], originally discovered in studies of thermodynamics in de-Sitter space, has recently been studied extensively in the context of developments in string theory, in particular in light of the role of holography in string theory [18]. The key point is that quantities such as the entropy and the energy do not scale extensively with the size of the volume, but increase as the area [18]. Thus, the average energy is

$$\langle E \rangle = TR^2 M_p^2, \quad (37)$$

which gives a special heat capacity

$$C_V(R) \sim R^2 M_p^2. \quad (38)$$

If we combine Eqs. (17), (19) and (38), and use the definition of the Gibbons-Hawking temperature associated with the instantaneous Hubble radius

$$T = \frac{1}{R}, \quad (39)$$

where R is taken to be the Hubble radius $R \sim 1/H$, then the power spectrum for Φ at the Hubble radius crossing is given by

$$\begin{aligned} P_\Phi(k, t_H(k)) &\sim \left(\frac{T_H(k)}{M_p}\right)^2 \\ &\sim \left(\frac{H(t_H(k))}{M_p}\right)^2 \sim k^{2/(1-p)}. \end{aligned} \quad (40)$$

Thus, a scale-invariant spectrum from the D-mode can be achieved in the limit $p \rightarrow \infty$, which corresponds to an inflationary contraction. However, in this case scales are not exiting the Hubble radius during the contracting phase, and thus it does not make sense to consider thermal fluctuations in this context since thermal equilibrium cannot be established on super-Hubble scales.

The contribution of the S-mode scales as

$$P_\Phi(k, t) \sim k^4 \eta_H(k)^{4\nu} k^{\frac{2p}{1-p} + 2}. \quad (41)$$

Inserting the relation (7) for ν in terms of p we find that the condition for scale-invariance is

$$w = 0. \quad (42)$$

As a consistency check, we note that, as follows from (40), for $p = 2/3$ the D-mode has a red spectrum. Hence, this is indeed a background equation of state for which thermal fluctuations of a holographic gas yields a scale-invariant spectrum.

C. Thermal Fluctuations of a String Gas

For completeness, we will also add an analysis for thermal string gas fluctuations. If space is compact, then the specific heat capacity of a gas of closed strings also satisfies the holographic scaling

$$C_V(R) \sim R^2 M_{pl}^2. \quad (43)$$

Inserting this relation into (18), (19) and (20) we find

$$P_\Phi(k, t_H(k)) \sim a^4(t_H(k)) T^2(t_H(k)). \quad (44)$$

In the case of string gas cosmology [6, 7, 9, 10], fluctuations exit the Hubble radius at the end of a quasi-static Hagedorn phase. Thus, both $a(t_H(k))$ and $T(t_H(k))$ are almost independent of k , and a scale-invariant spectrum (with a slight red-tilt) results.

If we forget about the Hagedorn background (which cannot be described in terms of the Einstein or dilaton gravity background equations), and simply couple a string gas to the non-singular background geometry discussed in this paper, and use $T(t) \sim a(t)^{-1}$, then the D-mode of Φ yields a scale-invariant spectrum if $p = 0$. This is consistent with the Hagedorn phase of string gas cosmology. We also find that the S-mode yields a scale-invariant contribution to the post-bounce spectrum of Φ if $p = 1/4$, but for this value of p , the contribution from the D-mode has a red tilt and hence dominates.

IV. NON-GAUSSIANITIES FROM THERMAL FLUCTUATIONS

Due to the non-linearities in the theory, the fluctuations are not perfectly Gaussian. A lot of recent interest has focused on calculating the non-Gaussianities as manifested in the three-point function (see e.g. [38] for a review). In single field slow-roll inflation models the amplitude of the predicted non-Gaussianities is suppressed by the slow-roll parameter. In models with a contracting phase, however, the induced non-Gaussianities are typically much larger (see e.g. [39] for studies in the context of the Ekpyrotic scenario).

As studied recently in [19], the non-Gaussianities as measured by the three-point function are of order one in the non-singular matter bounce scenario in which the fluctuations are of quantum vacuum origin. The key facts that lead to this result are firstly that there is no suppression of the non-Gaussianities by slow-roll parameters, and secondly that fluctuations grow on super-Hubble scales in the contracting phase, which results in a larger time interval determining the amplitude of the effects and in different terms dominating the shape of the three point function.

In this section we calculate the non-Gaussianity estimator f_{NL} in bouncing cosmologies with thermal fluctuations. We will use the formalism to compute non-Gaussianities of thermal origin which has been developed in [40] and which was applied to estimate the non-Gaussianities in string gas cosmology in [41] and to holographic cosmology in [42]. We will calculate the following non-Gaussianity estimator for fluctuations on a scale k :

$$f_{NL}(k) \equiv \frac{5}{18\sqrt{k^3}} \frac{\langle \zeta_k^3 \rangle}{\langle \zeta_k^2 \rangle^2}, \quad (45)$$

where ζ is the curvature fluctuation in co-moving gauge. Thus if we obtain the two-point and three-point correlators of metric perturbations originating from thermal fluctuations, the non-Gaussianity parameter can be calculated [47].

The expression for the two-point correlation function was given in (19). The three-point correlation function of thermal fluctuations in an equilibrium ensemble is

$$\langle \delta\rho^3 \rangle \equiv -\frac{1}{R^9} \frac{\partial^3 \ln Z}{\partial \beta^3} = \frac{T^2}{R^9} \frac{\partial}{\partial T} (C_R T^2), \quad (46)$$

where Z is the partition function and β is the inverse temperature.

A. Normal Radiation

For normal radiation with equation of state $w_r = 1/3$, we need the background equation of state of the universe in the contracting phase to be $w = 7/3$ in order to obtain a scale-invariant spectrum of fluctuations, as studied in Section 3. In this case, the heat capacity can be expressed as

$$C_V(R) = c_v R^3 T^3, \quad (47)$$

where c_v is determined by the background initial conditions and here is treated as a constant of order $O(1)$. From this, we obtain the following expressions for the two-point and three-point correlation functions of the density perturbations

$$\langle \delta\rho^2 \rangle = \frac{c_v T^5}{R^3}, \quad (48)$$

$$\langle \delta\rho^3 \rangle = \frac{5c_v T^6}{R^6}. \quad (49)$$

From Eq. (17) we find the relation

$$\Phi = \frac{\delta\rho}{2M_p^2 H^2}, \quad (50)$$

which is valid for any thermal system. In Section 3 we have seen that for the background considered here, the dominant mode after the bounce is seeded by the constant mode (the D-mode) in the contracting phase. Hence, the relation between Φ and ζ is given by

$$\zeta = \frac{5+3w}{3+3w} \Phi_D = \frac{6}{5} \Phi_D. \quad (51)$$

Combining the above equations, and going from Fourier space to position space via

$$\delta\rho = 2^{-1/2} \pi^{-1} \delta\rho(k), \quad (52)$$

where the left-hand side represents the root mean square density fluctuation corresponding to the co-moving scale k , finally the non-Gaussianity estimator takes the form

$$f_{NL} = \frac{25}{54\sqrt{2}\pi} M_p^2 H_{t_H(k)}^2 \frac{\langle \delta\rho^3 \rangle}{\langle \delta\rho^2 \rangle^2} \Big|_{t_H(k)}, \quad (53)$$

where the right hand side is evaluated at the Hubble radius crossing time. Inserting the above expressions for the density two and three point functions we get

$$f_{NL} = \frac{125}{54\sqrt{2}\pi} \frac{M_p^2 H_{t_H(k)}^2}{c_v T^4(t_H(k))}. \quad (54)$$

Noticing that for $w_r = 1/3$ we have $T \propto a^{-1}$, inserting $H(t_H(k)) \sim t^{-1}(t_H(k))$ and making use of (15) we obtain

$$f_{NL} \sim k^2 \frac{1-2p}{1-p}, \quad (55)$$

and the coefficient is such that for scales exiting the Hubble radius right before the bounce (when all quantities on the right hand side of (54) are of the order of the Planck mass - assuming that the energy scale at the bounce is given by the Planck mass) the amplitude of f_{NL} is of order 1. Inserting the value $p = 1/5$ obtained for $w = 7/3$ we finally obtain

$$f_{NL}(k) \sim \mathcal{O}(1) \left(\frac{k}{k_B}\right)^{3/2}, \quad (56)$$

where k_B is the value of k for which the wavelength exits the Hubble radius immediately before the bounce.

The above analysis shows that the non-Gaussianities in a bouncing cosmology in which the fluctuations are seeded by particle thermodynamic perturbations are Poisson suppressed on wavelengths larger than those exiting the Hubble radius immediately before the bounce. Thus, the non-Gaussianities are very different from what is obtained [19] in the case of a matter bounce with quantum vacuum initial perturbations. Besides the Poisson suppression of thermal non-Gaussianities (by the central limit theorem, on large scales the thermal fluctuations must approach a Gaussian), an important reason for the difference is that in the case of the “matter bounce” of [19] the S-mode is responsible for the final fluctuations. The S-mode grows on scales larger than the Hubble radius in the contracting phase, thus leading to an enhancement of the non-Gaussianities.

B. An Estimate of f_{NL} for Gibbons-Hawking Radiation

In the case of a holographic gas we expect the resulting non-Gaussianities to be much larger since for each scale the thermal correlation length is taken to be equal to the Hubble radius at the time that the scale exits the Hubble radius. Hence, we do not expect a Poisson suppression factor, and we expect non-Gaussianities to be of order 1. This expectation is verified by an explicit computation.

The starting point is the expression $C_V(R) = c_v M_p^2 R^2$ for the heat capacity, which leads to the following expression for the two-point and three-point correlation functions of energy density perturbations

$$\langle \delta\rho^2 \rangle = \frac{c_v M_p^2 T^2}{R^4}, \quad (57)$$

$$\langle \delta\rho^3 \rangle = \frac{2c_v M_p^2 T^3}{R^7}. \quad (58)$$

Inserting these expressions into (53) we obtain

$$f_{NL} = \frac{25}{54\sqrt{2}\pi} \frac{1}{c_v} \frac{RH^2}{T}, \quad (59)$$

where all quantities on the right hand side are to be evaluated at Hubble radius crossing.

Making use of the Gibbons-Hawking relation $T \simeq 1/R \simeq H$, we then have

$$f_{NL} \simeq \frac{1}{c_v}, \quad (60)$$

which shows the non-Gaussianity is of order $\mathcal{O}(1)$ and scale independent.

Note that the sign of f_{NL} is positive in the case of thermal fluctuations considered here. This is different from the negative sign obtained in the matter bounce scenario with vacuum initial conditions [19].

V. CONCLUSIONS

In this paper we have studied the possibility of obtaining a scale-invariant spectrum of fluctuations from thermal initial conditions in the context of a non-singular bouncing cosmology. We classified the conditions on the equation of state w_r of the thermal radiation and the equation of state w of the background which yield a scale-invariant spectrum.

In the case of regular particle radiation with $w_r = 1/3$ we find the condition $w = 7/3$ for the background. Such an equation of state can be realized by supposing that the background is determined by a scalar field with a negative exponential potential, similar to what is assumed in Ekpyrotic cosmology [37]. In this case, the fluctuations in radiation would be entropy fluctuations which would then seed adiabatic fluctuations with the same spectral index (note that the primordial adiabatic fluctuations induced by the scalar field are blue and hence will not dominate for long wavelength modes [25, 26]).

In the case of a Gibbons-Hawking radiation we find that an equation of state $w = 0$ (a matter bounce) yields a scale-invariant spectrum of cosmological perturbations.

Finally, to obtain a scale-invariant spectrum of cosmological perturbations with a string gas requires a quasi-static early phase. This matches with what is usually assumed in string gas cosmology [10].

We have also considered the non-Gaussianities of thermal fluctuations. In the case of thermal particle fluctuations we find that the non-Gaussianities are Poisson-suppressed on large scales. They are of the order 1 only on microscopic length scales, scales for which the thermal correlation length at the time of Hubble radius crossing is comparable with the Hubble radius itself. However, in the case of radiation satisfying the Gibbons-Hawking distribution, the non-Gaussianity is of order $1/c_v$ (where c_v is the constant which determines the specific heat capacity) and approximately scale independent.

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- [43] Another possibility for obtaining a scale-invariant spectrum of cosmological perturbations is in the context of string gas cosmology [6], in which it can be shown [7] that thermal string fluctuations in a quasi-static Hagedorn phase in the early universe lead to a scale-invariant spectrum of both scalar and tensor metric fluctuations, with a blue tilt of the tensor modes as a characteristic signature [8] (see [9, 10] for recent reviews).
- [44] We are assuming that the bounce is non-singular, as can be realized if the gravitational action has the specific higher derivative form given in [16] which is free of ghosts about Minkowski space-time and can be shown to lead to non-singular bounces given various forms of matter Lagrangians [16, 17]. We also assume that the bounce phase is short on the time scales of the fluctuations of interest. This assumption will be used when following the fluctuations through the bounce point.
- [45] We are only considering scalar metric fluctuations. However, it is important to point out that in cosmologies with a contracting phase, vector [21] and tensor [13, 22] fluctuations can also give a large contribution to the total angular power spectrum of cosmic microwave background anisotropies.
- [46] This calculation is rigorous in linear perturbation theory. However, its physical validity is only assured if the fluctuations remain in the linear regime throughout the evolution.
- [47] Implicitly, we are estimating the non-Gaussianities in the equilateral limit. In general, the three point function is a function of three momenta whose vector sum is zero. Here, we are taking the magnitudes of all momenta to be comparable, namely k .